

Precise bounds on the Higgs boson mass

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We study the renormalization group evolution of the Higgs quartic coupling λ_H and the Higgs mass m_H in the Standard Model. The one loop equation for λ_H is non linear and it is of the Riccati type which we numerically and analytically solve in the energy range $[m_t, E_{GU}]$ where m_t is the mass of the top quark and $E_{GU} = 10^{14}$ GeV. We find that depending on the value of $\lambda_H(m_t)$ the solution for $\lambda_H(E)$ may have singularities or zeros and become negative in the former energy range so the ultra violet cut off of the standard model should be below the energy where the zero or singularity of λ_H occurs. We find that for $0.369 \leq \lambda_H(m_t) \leq 0.613$ the Standard Model is valid in the whole range $[m_t, E_{GU}]$. We consider two cases of the Higgs mass relation to the parameters of the standard model: (a) the effective potential method and (b) the tree level mass relations. The limits for $\lambda_H(m_t)$ correspond to the following Higgs mass relation $150 \leq m_H \lesssim 193$ GeV. We also plot the dependence of the ultra violet cut off on the value of the Higgs mass. We analyze the evolution of the vacuum expectation value of the Higgs field and show that it depends on the value of the Higgs mass. The pattern of the energy behavior of the VEV is different for the cases (a) and (b). The behavior of $\lambda_H(E)$, $m_H(E)$ and $v(E)$ indicates the existence of a phase transition in the standard model. For the effective potential this phase transition occurs at the mass range $m_H \approx 180$ GeV and for the tree level mass relations at $m_H \approx 168$ GeV.

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I. INTRODUCTION

The Standard Model (SM) provides a very precise description of all the present elementary particle data [1]. On the other hand it has relatively many free parameters (~ 19) what is rather unsatisfactory from the fundamental point of view. The idea of Grand Unification (GU) [2] is to look for additional symmetries in the SM at very high energies. The most notable sign of the presence of GU is the (approximate) convergence of the three gauge couplings to one common value at the energies $10^{14} - 10^{15}$ GeV. This allows to substitute the gauge group $SU(3) \times SU(2) \times U(1)$ of the SM by a larger group and to reduce the number of gauge couplings to only one.

The main tool of the GU models are the Renormalization Group Equations (RGE) [3] that relate various observables (like couplings or masses) at different energies and also allow the study of their asymptotic behavior.

In the perturbative quantum field theory the RGE are differential equations for the observables which are obtained from the condition that the S -matrix elements do not depend on the renormalization scheme or renormalization point. The right hand side of the RGE is an infinite series expanded according to the number of loops. Most of the numerical and analytical results for the RGE [4] are to the order of one loop only possibly with a partial inclusion of two loops, while the RGE for most of the observables have been given for the SM and its extensions up to two loops [4, 5, 6].

The right hand side of the RGE is constructed from the following terms

$$g_i^2, \quad y_u y_u^\dagger, \quad y_d y_d^\dagger, \quad y_l y_l^\dagger, \quad y_\nu y_\nu^\dagger, \quad \lambda_H, \quad (1)$$

where the g_i 's are the gauge couplings, y_u, y_d, y_l, y_ν , are the Yukawa couplings of the up and down quarks, charged leptons and neutrinos, respectively and λ_H is the Higgs quartic coupling constant. The RGE form a set of non-linear coupled differential equations and even at the one loop order there exist only approximate or numerical solutions [4, 7].

The one loop RGE for the best measured observables g_i 's, quark and lepton Yukawa couplings and the Cabibbo-Kobayashi-Maskawa (CKM) matrix are independent of the Higgs quartic coupling. This allows to derive the running

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of those observables at the lowest order without the knowledge of the λ_H . On the other hand, at the two loop level, the quartic coupling λ_H appears in the RGE for many observables like quark masses or the CKM matrix and has important influence on their behavior and cannot be neglected.

The one loop equation for λ_H is also non linear and has been used to obtain the limits on the Higgs mass from the triviality of the $\lambda\phi^4$ theory and the existence of the Landau pole. This equation has been also considered in Refs. [8, 9, 10] to study the dependence of the Higgs mass and the UV cut off on the energy and it was solved for the simplified case when the gauge couplings and the top quark Yukawa coupling are constant.

In this paper we study the one loop equation for λ_H without any simplifying assumptions in the energy range, starting at the top quark mass m_t . We find that the equation is of the Riccati type and we solve this equation explicitly. We find that for the values of λ_H at the top quark mass, $\lambda_H(m_t) \geq 0.528$, the function $\lambda_H(E)$ has a Landau singularity. For the values of $\lambda_H(m_t) \leq 0.386$ there is no Landau pole below the energies E_{GU} and the solution $\lambda_H(E)$ passes through zero and then becomes negative. This means that for the latter values of $\lambda_H(m_t)$ the theory becomes unstable and the UV cutoff should appear below the energy value corresponding to the zero of $\lambda_H(E)$. As is well known the coupling λ_H is related to the Higgs mass so our results are presented in terms of the Higgs mass.

We also study the one loop RGE for the Higgs mass m_H . We consider two cases here. The first one is where the Higgs mass is obtained from the Higgs field effective potential and in the other one the tree level relation of the Higgs mass with the parameters of the SM Lagrangian is used. We explicitly solve these equations for both cases and use them to find the UV cut off of the standard model and show the evolution of the Higgs mass in the range $120 \leq m_H \leq 500$ GeV. We next use the evolution of the m_H and λ_H to find the evolution of the vacuum expectation value of the Higgs field. All these results show that there is a sharp transition in the behavior of the standard model at the Higgs mass $m_H \approx 180$ GeV for the case of the effective potential and at $m_H \approx 168$ GeV for the tree level relations. Additionally we show that for the case of the effective potential for the Higgs mass in the range $150 \leq m_H \leq 193$ GeV the standard model is valid up to the GU energy $\sim 10^{14}$ GeV. For the case of the tree level mass relations the former limit is $150 \leq m_H \leq 194$ GeV.

II. ONE AND TWO LOOPS RENORMALIZATION GROUP EQUATIONS

The two loop RGE are the following

$$\frac{dg_l}{dt} = \frac{1}{(4\pi)^2} b_l g_l^3 - \frac{1}{(4\pi)^4} G_l g_l^3, \quad (2a)$$

$$\frac{dy_{u,d,e,\nu}}{dt} = \left[\frac{1}{(4\pi)^2} \beta_{u,d,e,\nu}^{(1)} + \frac{1}{(4\pi)^4} \beta_{u,d,e,\nu}^{(2)} \right] y_{u,d,e,\nu}, \quad (2b)$$

$$\frac{d\lambda_H}{dt} = \left[\frac{1}{(4\pi)^2} \beta_\lambda^{(1)} + \frac{1}{(4\pi)^4} \beta_\lambda^{(2)} \right], \quad (2c)$$

$$\frac{dm^2}{dt} = \left[\frac{1}{(4\pi)^2} \beta_{m^2}^{(1)} + \frac{1}{(4\pi)^4} \beta_{m^2}^{(2)} \right]. \quad (2d)$$

where $t = \ln(E/m_t)$, E is the energy and $m_t = 174.1$ GeV is the top quark mass and the Higgs potential is $m^2 \phi^\dagger \phi + (\lambda_H/2)(\phi^\dagger \phi)^2$. The constants b_l depend on the model and G_l , $\beta_{u,d,e,\nu}^{(1)}$, $\beta_{u,d,e,\nu}^{(2)}$, $\beta_\lambda^{(1)}$, $\beta_\lambda^{(2)}$, $\beta_{m^2}^{(1)}$, $\beta_{m^2}^{(2)}$ are functions of the standard model couplings and the squares of the Yukawa couplings $H_{u,d,e,\nu}^{(1)} = y_{u,d,e,\nu} y_{u,d,e,\nu}^\dagger$, (for the definition of these functions and constants see [4] or [7]).

In the previous papers [7] we have discussed a consistent approximation scheme for the solution of the RGE that was based on the expansion of the solutions in terms of the powers of λ , where $\lambda \simeq 0.22$ is the absolute value of the $|V_{us}|$ element of the CKM matrix.

In such an approximation the lowest order RGE have the following form [5, 6] [18]

$$\frac{dg_i}{dt} = \frac{1}{(4\pi)^2} b_i g_i^3, \quad i = 1, 2, 3, \quad (3a)$$

$$\frac{dy_u}{dt} = \frac{1}{(4\pi)^2} \{ \alpha_1^u(t) + \alpha_2^u y_u y_u^\dagger + \alpha_3^u \text{Tr}(y_u y_u^\dagger) \} y_u, \quad (3b)$$

$$\frac{dy_d}{dt} = \frac{1}{(4\pi)^2} \{ \alpha_1^d(t) + \alpha_2^d y_u y_u^\dagger + \alpha_3^d \text{Tr}(y_u y_u^\dagger) \} y_d, \quad (3c)$$

$$\frac{d\lambda_H}{dt} = \frac{12}{(4\pi)^2} \left\{ \lambda_H^2 + \left[\text{Tr}(y_u y_u^\dagger) - \frac{3}{4} \left(\frac{1}{5} g_1^2 + g_2^2 \right) \right] \lambda_H + \frac{3}{16} \left(\frac{3}{25} g_1^4 + \frac{2}{5} g_1^2 g_2^2 + g_2^4 \right) - \text{Tr}(y_u y_u^\dagger)^2 \right\}, \quad (3d)$$

$$\frac{d \ln m^2}{dt} = \frac{1}{(4\pi)^2} \left\{ 6\lambda_H + 6 \text{Tr}(y_u y_u^\dagger) - \frac{9}{2} \left(\frac{1}{5} g_1^2 + g_2^2 \right) \right\}. \quad (3e)$$

The constants b_i and α 's in Eqs. (3) are equal

$$\begin{aligned} (b_1, b_2, b_3) &= \left(\frac{41}{10}, -\frac{19}{6}, -7 \right) \\ \alpha_1^u(t) &= - \left(\frac{17}{20} g_1^2 + \frac{9}{4} g_2^2 + 8g_3^2 \right), & \alpha_2^u &= \frac{3}{2} b, & \alpha_3^u &= 3, \\ \alpha_1^d(t) &= - \left(\frac{1}{4} g_1^2 + \frac{9}{4} g_2^2 + 8g_3^2 \right), & \alpha_2^d &= \frac{3}{2} c, & \alpha_3^d &= 3a, \\ (a, b, c) &= (1, 1, -1). \end{aligned}$$

Eqs. (3a)-(3c) can be *explicitly* solved and the most important results and properties of these solutions are [7]

1. g_i 's, y_u and y_d are all regular functions of energy in the range $[m_t, E_{GU}]$.
2. The running of the gauge couplings $g_i(t)$ is $(t_0 = \ln(E/m_t)|_{E=m_t} = 0)$

$$(g_i(t))^2 = \frac{(g_i(t_0))^2}{1 - \frac{2}{(4\pi)^2} (g_i(t_0))^2 b_i(t - t_0)}. \quad (4)$$

3. The running of the up quark Higgs couplings $y_u(t)$ has the following property

$$\text{Tr}(y_u y_u^\dagger) = Y_t^2(t) = \frac{Y_t^2(t_0) r(t)}{1 - \frac{2(\alpha_2^u + \alpha_3^u)}{(4\pi)^2} Y_t^2(t_0) \int_{t_0}^t r(\tau) d\tau} \quad (5)$$

where Y_t is the largest eigenvalue of the up quark Higgs coupling matrix y_u and $r(t) = \exp \left((2/(4\pi)^2) \int_{t_0}^t \alpha_1^u(\tau) d\tau \right) = \prod_{k=1}^3 [g_k^2(t_0)/g_k^2(t)]^{c_k/b_k}$, $c_k = (17/20, 9/4, 8)$.

Using Eqs. (4) and (5) as input into Eq. (3d) we obtain the uncoupled differential equation for the quartic coupling constant λ_H .

Eq. (3d) for λ_H has been considered earlier by various authors [8, 9, 10] but in all these papers the effects of the running of the gauge couplings and of Y_t^2 have not been considered. The importance of λ_H for the evolution of other observables comes from the fact that λ_H appears at the two loop order in the RGE for y_u and y_d and at the one loop order for m_H .

III. ONE LOOP EQUATION FOR λ_H

The one loop equation for λ_H given in Eq. (3d) is rewritten in the form

$$\frac{d\lambda_H}{dt} = f_0(t) + f_1(t)\lambda_H + f_2(t)\lambda_H^2, \quad (6)$$

where the definition of the functions $f_i(t)$ can be deduced from Eq. (3d). This equation is of the Riccati type [11]. The behavior of the gauge coupling g_i 's is given in Eq. (4) and the explicit energy dependence of $\text{Tr}(y_u y_u^\dagger)$ is given in Eq. (5). As discussed before the g_i 's and $\text{Tr}(y_u y_u^\dagger)$, as functions of energy, have no singularities in the range $[m_t, E_{GU}]$. On the other hand the solutions of the Riccati's equations can become singular even if the coefficients of the equation are smooth and regular functions.

The solution of Eq. (6) is obtained by substituting the λ_H by the following expression containing the auxiliary function $W(t)$

$$\lambda_H(t) = -\frac{1}{f_2(t)} \frac{W'(t)}{W(t)} \quad (7)$$

which fulfills the linear second order differential equation

$$W'' - \left(\frac{f_2'(t)}{f_2(t)} + f_1(t) \right) W' + f_0(t) f_2(t) W = 0. \quad (8)$$

Any solution of Eq. (8) generates the solutions of Eq. (6). Eq. (8) is of the Frobenius type [12] and the solution $W(t)$ is a *regular* function of the energy t in the region where the coefficients of Eq. (8) are *regular*. One can look for the solutions of this equation in terms of an infinite series. We look for the two solutions of this equation with the following properties

$$\begin{aligned} W_1(t)|_{t_0} &= 1, & W_1'(t)|_{t_0} &= 0, \\ W_2(t)|_{t_0} &= 0, & W_2'(t)|_{t_0} &= 1. \end{aligned} \quad (9)$$

The solution of (6) for λ_H in terms of the functions $W_1(t)$ and $W_2(t)$ has the following form (note that $f_2(t) = 12/(4\pi)^2$)

$$\lambda_H(t) = -\frac{(4\pi)^2}{12} \frac{W_1'(t) - \frac{12}{(4\pi)^2} \lambda_H(t_0) W_2'(t)}{W_1(t) - \frac{12}{(4\pi)^2} \lambda_H(t_0) W_2(t)}. \quad (10)$$

The most important property of the solution (10) is that the singularities of the solution $\lambda_H(t)$ are determined from the zeros of the denominator

$$W_1(t) - \frac{12}{(4\pi)^2} \lambda_H(t_0) W_2(t) = 0 \quad (11)$$

and the zeros of $\lambda_H(t)$ are determined from the zeros of the numerator

$$W_1'(t) - \frac{12}{(4\pi)^2} \lambda_H(t_0) W_2'(t) = 0 \quad (12)$$

then one can precisely determine the position of the singularities and zeros and their dependence on the initial value of the Higgs quartic coupling $\lambda_H(t_0)$. The detailed discussion of the solutions is given in the next section.

IV. RUNNING OF λ_H

In this section we will discuss the explicit solutions of Eqs. (8) and (3d). Let us start with Eq. (8). The form of the functions $-(f_2'(t)/f_2(t) + f_1(t))$ and $f_0(t)f_2(t)$ is too complicated to be able to solve Eq. (8) explicitly. To find the solution of this equation we use the fact that they are smooth functions of energy so we approximate these two functions in the energy range $[m_t, E_{GU}]$ by the ratio of two polynomials. These functions perfectly approximate both coefficients in Eq. (8) in the whole energy range and this allows to find the solution of Eq. (8) in terms of a power series of the variable t [19]. In Fig. 1 we show the dependence on the energy of the two solutions of Eq. (8) and their derivatives [20]. As expected they are smooth functions of t .

From Eq. (10) we find now the dependence of $\lambda_H(t)$ on the energy t and important properties of its behavior. It is the most interesting to investigate how $\lambda_H(t)$ depends on the initial values of $\lambda_H(t_0)$ and to find out the range of validity of the SM. As discussed earlier, for the SM to be valid $\lambda_H(t)$ must be positive and cannot be singular. Since $\lambda_H(t_0) > 0$, it means that the SM is valid for energies between m_t and the zero or singularity of $\lambda_H(t)$ which can be determined from Eqs. (11) and (12).

Let us first consider the singularity (a simple pole) of $\lambda_H(t)$. For this purpose we plot in Fig. 2 the ratio of the two solutions $(12/(4\pi)^2)W_2(t)/W_1(t)$ from which we can determine the value of t for which the pole occurs depending on the value of $\lambda_H(t_0)$. If we impose the condition that $\lambda_H(t)$ is regular in the whole range of the energies $[m_t, E_{GU}]$ then the value of $1/\lambda_H(t_0)$ should lie above the curve in Fig. 2 what gives the following condition

$$\lambda_H(t_0) \leq 0.613. \quad (13)$$

For the SM to be valid the quartic coupling $\lambda_H(t)$ should not become negative. We use Eq. (12) to find the first zero of $\lambda_H(t)$. In Fig. 3 we have plotted $((4\pi)^2/12)W_1'(t)/W_2'(t)$ which determines at which energy in t occurs the first zero of $\lambda_H(t)$ depending on the value of $\lambda_H(t_0)$. Now from the condition that $\lambda_H(t)$ should not have zeros in the whole range of the energies $[m_t, E_{GU}]$ we obtain

$$\lambda_H(t_0) \geq 0.369. \quad (14)$$

We thus see that the consistency of the SM in the range of the energies up to the grand unification energy E_{GU} permits a very narrow band on the admissible values of the $\lambda_H(t_0)$.

$$0.369 \leq \lambda_H(t_0) \leq 0.613. \quad (15)$$

In the next section we discuss the implications of the running of λ_H for the Higgs mass and the VEV of the Higgs field.

V. RUNNING OF THE HIGGS MASS AND VEV

The most interesting predictions from the one loop *exact* solution for λ_H can be obtained for the Higgs boson mass. This problem has received earlier a wide attention [13] and we will discuss here the method based on the Higgs effective potential [14, 15]. Recently there also appeared a series of papers [17] in which the full two loop *analytical* analysis of the pole masses of the gauge bosons has been made. It has been shown there that the tree level relations between the pole masses and the couplings of the standard model are valid up to two loops.

We will compare the predictions of these two approaches for the running of the Higgs field mass and VEV.

A. Effective potential method

The square of the Higgs boson mass is defined as the second derivative of the effective potential taken at the minimum of the potential [16]. We will use the following form of the effective potential [5, Eq. (13)]

$$V_{\text{eff}} = m^2(t)Z^2(t)\phi^\dagger\phi + \frac{1}{2}\lambda_H(t)Z^4(t)(\phi^\dagger\phi)^2 \quad (16)$$

where $Z(t)$ is the renormalization factor of the Higgs field that fulfills the RG equation [14, Eq. (10)]

$$\frac{d \ln Z}{dt} = -\gamma_\phi(g_i^2, Y_t^2) = \frac{3}{(4\pi)^2} \left(\frac{3}{20}g_1^2 + \frac{3}{4}g_2^2 - Y_t^2 \right) \quad (17)$$

which has the solution

$$Z(t) = h_m^{-3}(t) \left(\frac{g_1(t)}{g_1(t_0)} \right)^{\frac{9}{20b_1}} \left(\frac{g_2(t)}{g_2(t_0)} \right)^{\frac{9}{4b_2}} \quad (18)$$

and $h_m(t)$ is equal [7]

$$h_m = \exp \left(\frac{1}{(4\pi)^2} \int_{t_0}^t \text{Tr}(y_u y_u^\dagger) dt \right). \quad (19)$$

From Eq. (16) we obtain the following result for the physical Higgs mass

$$m_H^2(t) = -2m^2(t)Z^2(t) \quad (20)$$

and now combining Eqs. (3e) and (17) we obtain the following RG equation for the physical Higgs mass

$$\frac{d \ln m_H^2}{dt} = \frac{6}{(4\pi)^2} \lambda_H. \quad (21)$$

which has the following simple solution

$$m_H^2(t) = \frac{m_H^2(t_0)}{\left(W_1(t) - \frac{12}{(4\pi)^2} \lambda_H(t_0) W_2(t)\right)^{1/2}} \quad (22)$$

The Higgs field vacuum expectation value can also be calculated from the effective potential (16) and is equal

$$v^2(t) = \frac{m_H^2(t)}{2\lambda_H(t)Z^4(t)} = -m_H^2(t_0) \frac{6}{(4\pi)^2} \frac{\left(W_1(t) - \frac{12}{(4\pi)^2} \lambda_H(t_0) W_2(t)\right)^{1/2}}{W_1'(t) - \frac{12}{(4\pi)^2} \lambda_H(t_0) W_2'(t)} h_m^{12}(t) \left(\frac{g_1(t_0)}{g_1(t)}\right)^{\frac{9}{5b_1}} \left(\frac{g_2(t_0)}{g_2(t)}\right)^{\frac{9}{b_2}}. \quad (23)$$

We thus see that $m_H^4(t)$ and $\lambda_H(t)$ have a pole in the same position. $v^2(t)$ has a pole where $\lambda_H(t)$ has a zero and $v^2(t)$ has a zero at the position of the Landau pole.

B. Tree level relations

In the case of the tree level relations the pole mass and the VEV of the Higgs field are given by the following simple relations [17]

$$m_H^2(t) = -2m^2(t), \quad v^2(t) = \frac{m_H^2(t)}{2\lambda_H(t)}. \quad (24)$$

From previous equation it follows that the RG equation for the Higgs mass is the same as for the parameter m^2 of the Higgs potential

$$\frac{d \ln m_H^2}{dt} = \frac{1}{(4\pi)^2} \left\{ 6\lambda_H + 6 \text{Tr}(y_u y_u^\dagger) - \frac{9}{2} \left(\frac{1}{5} g_1^2 + g_2^2 \right) \right\}. \quad (25)$$

which has the following solution

$$m_H^2(t) = \frac{m_H^2(t_0)}{\left(W_1(t) - \frac{12}{(4\pi)^2} \lambda_H(t_0) W_2(t)\right)^{1/2}} h_m^6(t) \left(\frac{g_1(t_0)}{g_1(t)}\right)^{\frac{9}{10b_1}} \left(\frac{g_2(t_0)}{g_2(t)}\right)^{\frac{9}{2b_2}} \quad (26)$$

and the VEV of the Higgs field is equal

$$v^2(t) = \frac{m_H^2(t)}{2\lambda_H(t)} = -m_H^2(t_0) \frac{6}{(4\pi)^2} \frac{\left(W_1(t) - \frac{12}{(4\pi)^2} \lambda_H(t_0) W_2(t)\right)^{1/2}}{W_1'(t) - \frac{12}{(4\pi)^2} \lambda_H(t_0) W_2'(t)} h_m^6(t) \left(\frac{g_1(t_0)}{g_1(t)}\right)^{\frac{9}{10b_1}} \left(\frac{g_2(t_0)}{g_2(t)}\right)^{\frac{9}{2b_2}}. \quad (27)$$

The analytical properties of the Higgs mass and the VEV of the Higgs field are similar to the previous case.

VI. CONCLUSIONS

The numerical predictions for both cases are presented in Figs. 4, 5, 6 and 7. These figures contain the upper value for the UV cut off as a function of the Higgs mass, running of the Higgs mass, VEV of the Higgs field and of $\lambda_H(t)$. To obtain these figures we used the Higgs boson matching scale equal to $m_H = \max\{m_t, m_H(t_H)\}$, where $t_H = \ln(m_H/m_t)$. The figures 4a, 5a, 6a, 7a correspond to the case of the effective potential and the figures 4b, 5b, 6b, 7b correspond to the case of the tree level case.

The left hand side of Fig. 4 consists of only one curve that is obtained from the condition $\lambda_H(t) = 0$. For the Higgs masses that allow this condition there is no Landau pole up to the GU energy E_{GU} and the values of λ_H are small and the function $\lambda_H(t)$ is monotonically decreasing (see Fig. 7) so the two loop and higher corrections are small and the

perturbation series does not diverge. The right hand side of the figure consists of the three curves. The upper curve corresponds to the position of the Landau pole. Since it is well known that for the energies close the position of the Landau pole the perturbation series breaks down and the two and higher order loop corrections become important we have drawn two additional, more realistic curves that correspond to the values of λ_H when the two and three loops corrections become important [9, Eq. (3.2), $\lambda_{FP} \approx 12.1$]: the lower curve corresponds to $\lambda_H = \lambda_{FP}/4$ and the middle curve to $\lambda_H = \lambda_{FP}/2$. These curves differ significantly from the one for the Landau pole curve only for the high Higgs masses.

From Fig. 4 we see that for the Higgs masses $m_H \leq 152$ GeV the UV cut off is growing as a function of the Higgs mass. The energy of grand unification E_{GU} is reached at $m_H \approx 150$ GeV and then there is a narrow window of the Higgs masses

$$\begin{aligned} 150 \leq m_H \leq 193 \text{ GeV} & \quad \text{for the effective potential,} \\ 150 \leq m_H \leq 194 \text{ GeV} & \quad \text{for the tree level mass relations,} \end{aligned} \quad (28)$$

for which the UV cut off exceeds the E_{GU} scale. For the Higgs masses $m_H \geq 178$ there appears the Landau pole and the UV cut off is decreasing as a function of the Higgs mass. Here, the most realistic is the lowest curve in Fig. 4 which corresponds to the point where the perturbation series ceases to be meaningful.

In Fig. 5 we show the evolution of the Higgs boson mass in the range $120 \leq m_H \leq 500$ GeV. We see that according to the initial Higgs mass there are two distinct patterns of evolution. In case (a) of the effective potential for $m_H < 174$ GeV there is a slow linear evolution up to the UV cut off. For $m_H > 178$ GeV there is a singularity at the UV cut off and the growth of $m_H(t)$ is much faster especially for high Higgs boson masses. In case (b) of the tree level mass relations for $m_H < 164$ GeV there is a slow linear evolution up to the UV cut off. For $m_H > 178$ GeV there is a singularity at the UV cut off and the growth of $m_H(t)$ is much faster especially for high Higgs boson masses. The behavior of $m_H(t)$ is significantly different for both cases.

In Fig. 6 we show the evolution of the VEV of the Higgs field derived from Eq. (23) for the Higgs masses in the range $120 \leq m_H \leq 500$ GeV. Similarly as in the case of the Higgs mass there are two patterns of evolution. In case (a) of the effective potential for the Higgs masses $m_H < 174$ GeV the VEV is a growing function of energy and has a singularity at the UV cut off. For the Higgs masses $m_H > 182$ GeV the VEV initially increases with energy then bends and hits zero at the UV cut off. In case (b) of the tree level mass relations for the Higgs masses $m_H < 164$ GeV the VEV is a growing function of energy and has a singularity at the UV cut off. For the Higgs masses $m_H > 178$ GeV the VEV decreases with energy and hits zero at the UV cut off. The behavior of $v(t)$ is significantly different for both cases.

In Fig. 7 we show the evolution of the coupling $\lambda_H(t)$ given in Eq. (10). One can see that the behavior of $\lambda_H(t)$ is in agreement with the earlier discussion and for values of $\lambda_H(t_0) < 0.37$ the function $\lambda_H(t)$ has a zero and for $\lambda_H(t_0) > 0.61$ it has a pole. The dependence of $\lambda_H(t)$ on the Higgs mass for the case (a) of the effective potential and (b) of the tree level mass relations is very similar.

The behavior of the Higgs VEV has important implications for the standard model. It means that the masses of the gauge bosons and quarks would grow with energy for $m_H < 174$ GeV. For $182 < m_H < 300$ GeV the masses of the gauge bosons and quarks would first increase with energy, then reach a maximum and decrease. For the Higgs masses $m_H > 300$ GeV the VEV rapidly tends to zero.

From the evolution of the Higgs mass and VEV one can see the appearance of two patterns of the high energy behavior of the standard model. One, for $m_H < 174$ GeV and the other for $m_H > 182$ GeV. The Higgs with $m_H = 180$ GeV is the point of the transition between the two patterns for the case of the effective potential and $m_H = 168$ GeV is the transition point for the tree level mass relations.

It is interesting that both cases: the Higgs effective potential and the tree level mass relations give very similar results for the Higgs mass limits (see Fig. 4) while the energy behavior of the VEV and $m_H(t)$ is significantly different. This has the origin in the fact that the position of the UV cut off obtained from Eqs. (11) and (12) is insensitive to the running of the Higgs mass derived from Eqs. (22) and (26).

The results of this paper support the results of Refs. [9, 10] where the similar problem was considered. In Ref. [9] the authors were using the simplified assumption that the gauge couplings and the top quark Yukawa coupling are constant and do not run according to the RGE. Our treatment is more precise and the simplifying assumptions are not necessary. It is interesting that the running of the gauge couplings and the top quark Yukawa coupling have an important influence on the results especially for the low Higgs masses, where the two loop corrections are negligible.

To conclude let us stress that the key point of the paper is the treatment of the one loop RG equation for λ_H and the linearization of the problem by the substitution in Eq. (7). This linearization permitted very precise analysis of the positions of the Landau pole and the point where λ_H vanishes. Moreover it also gave an intimate relation between the positions of these two points: Eq. (12) is the derivative of Eq. (11). Such a relation between these two important quantities is a new result. Additionally it should be also stressed that the *analytical* results for running of λ_H up to one loop is a very good starting for a precise analysis of the two loop effects.

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 - [18] We neglect the leptonic part of the RGE because it decouples from the quark sector in the considered approximation.
 - [19] We applied two methods of the solution of Eq. (8), one based on the explicit power expansions and the other using Mathematica capabilities of the numerical solution of the differential equations. Both approaches agree perfectly.
 - [20] In the numerical calculations we use the following initial values of the parameters: $g_1(t_0) = 0.4459$, $g_2(t_0) = 0.62947$, $g_3(t_0) = 1.2136$, $m_t(t_0) = 174.1$ and $v(t_0) = 174.1$

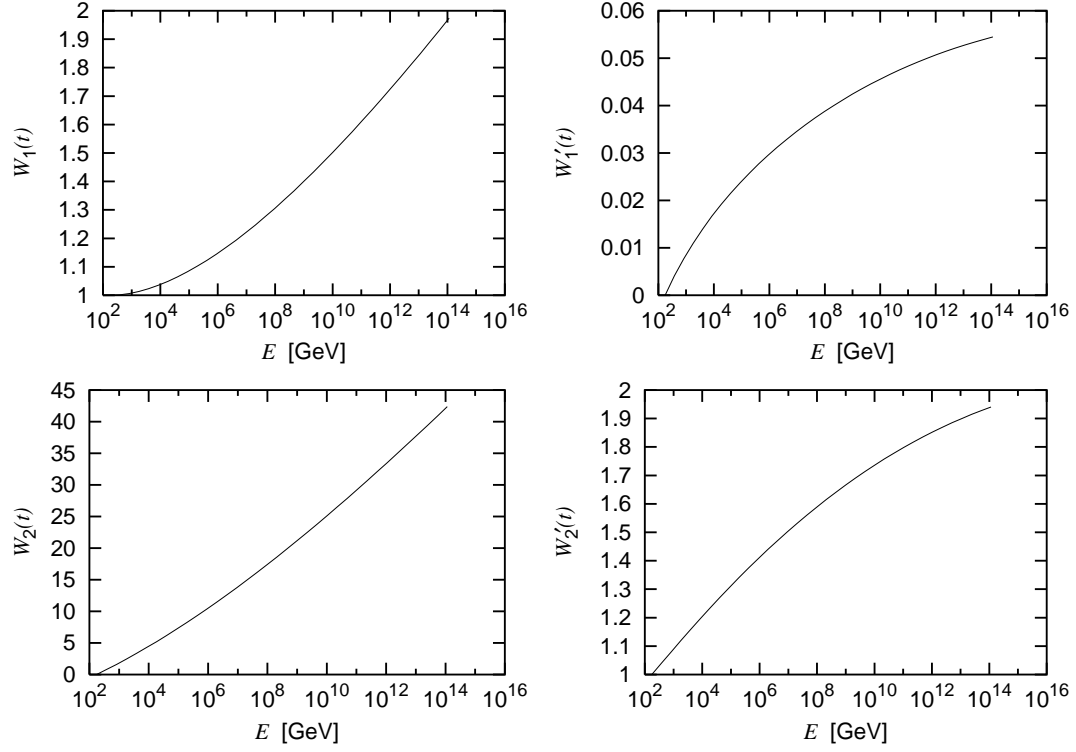


FIG. 1: The solutions of Eq (8) and their derivatives.

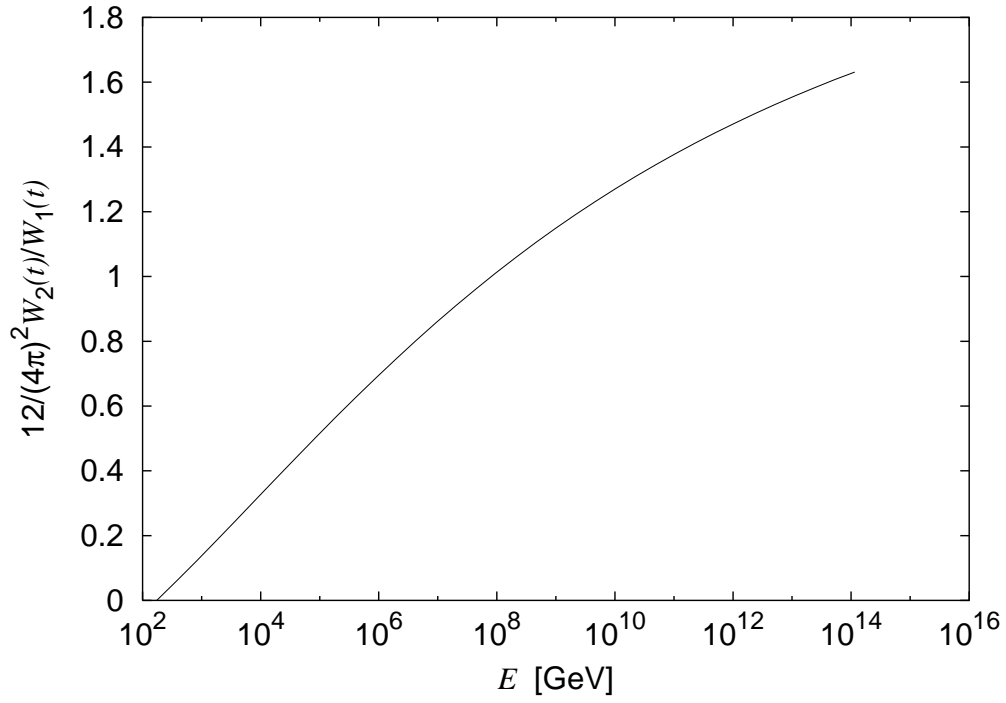


FIG. 2: The ratio of the solutions $(12/(4\pi)^2)W_2(t)/W_1(t)$ of Eq(8)). This ratio determines the value of t at which $1/\lambda_H(t)$ vanishes, i.e. $\lambda_H(t)$ and $m_H^4(t)$ have a pole and $v^4(t)$ has a zero.

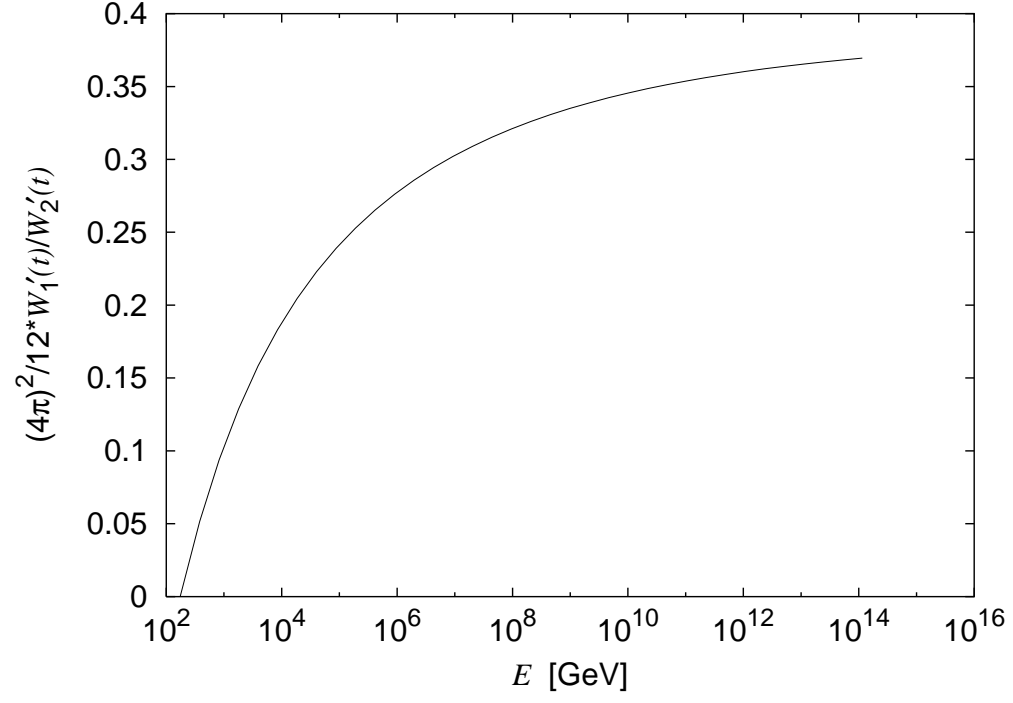


FIG. 3: The ratio of the derivatives $((4\pi)^2/12)W_1'(t)/W_2'(t)$ of the solutions of Eq((8)). This ratio determines the position t at which $\lambda_H(t)$ has a zero and $v^2(t)$ has a pole.

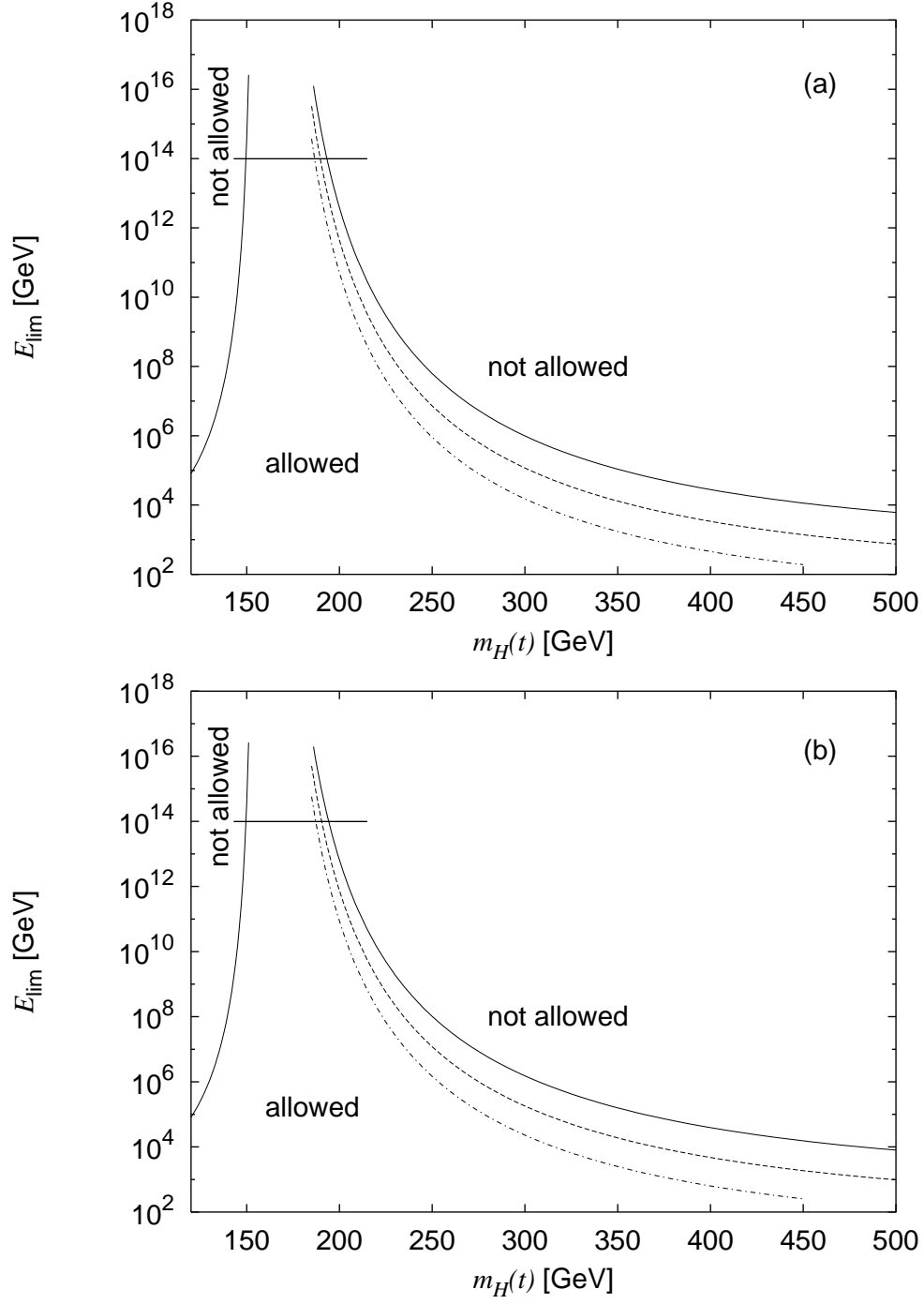


FIG. 4: The plot of the energy E_{lim} at which the SM breaks down as the function of the Higgs mass. Figure (a) corresponds to the case of the effective potential and figure (b) to the tree-level relation for the Higgs mass. The curve on the left hand side is derived from the vanishing of $\lambda_H(t)$. The three curves on the right hand side are obtained from the condition that the perturbation expansion of the renormalization group equation for λ_H breaks down: the continuous line follows from the position of the Landau pole, the middle curve corresponds to the value $\lambda_H = \lambda_{FP}/2$ and the lowest one to $\lambda_H = \lambda_{FP}/4$ ($\lambda_{FP} \approx 12.1$ is the fixed point value of the two loop β function for λ_H [9]). Both cases (a) and (b) give very similar form for the Higgs mass limits.

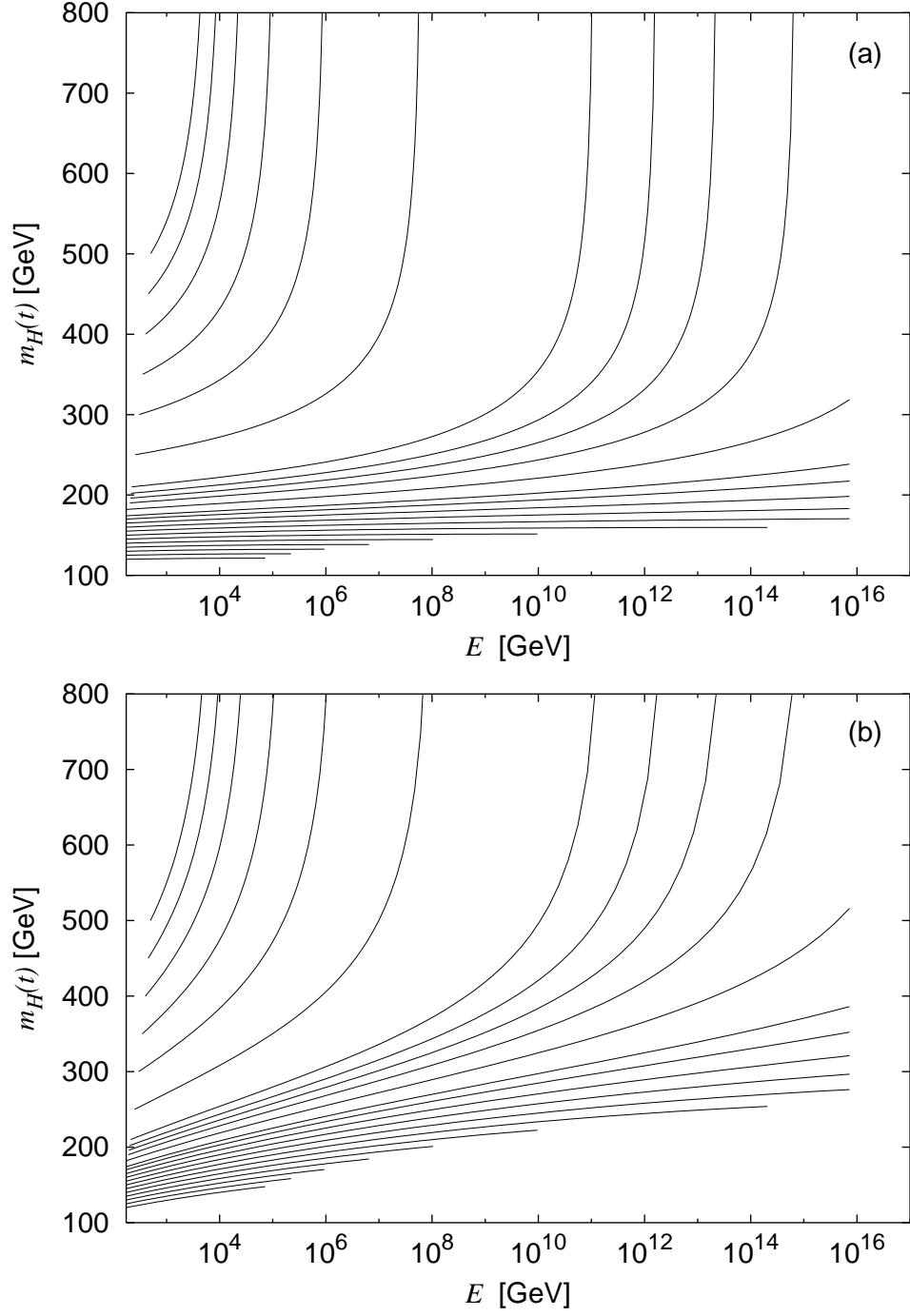


FIG. 5: Running of the Higgs mass. Figure (a) corresponds to the case of the effective potential and figure (b) to the tree-level relation for the Higgs mass. In case (a) for the Higgs masses $m_H < 174$ GeV the function $m_H(t)$ has no singularity and is almost linear. For Higgs masses $m_H > 190$ GeV $m_H(t)$ has a singularity (a pole) which defines the upper limit for the UV cut off. In case (b) for the Higgs masses $m_H < 164$ GeV the function $m_H(t)$ has no singularity and is almost linear. For Higgs masses $m_H > 178$ GeV $m_H(t)$ has a singularity (a pole). In both cases for the Higgs masses ~ 500 GeV $m_H(t)$ is a very steep function of t .

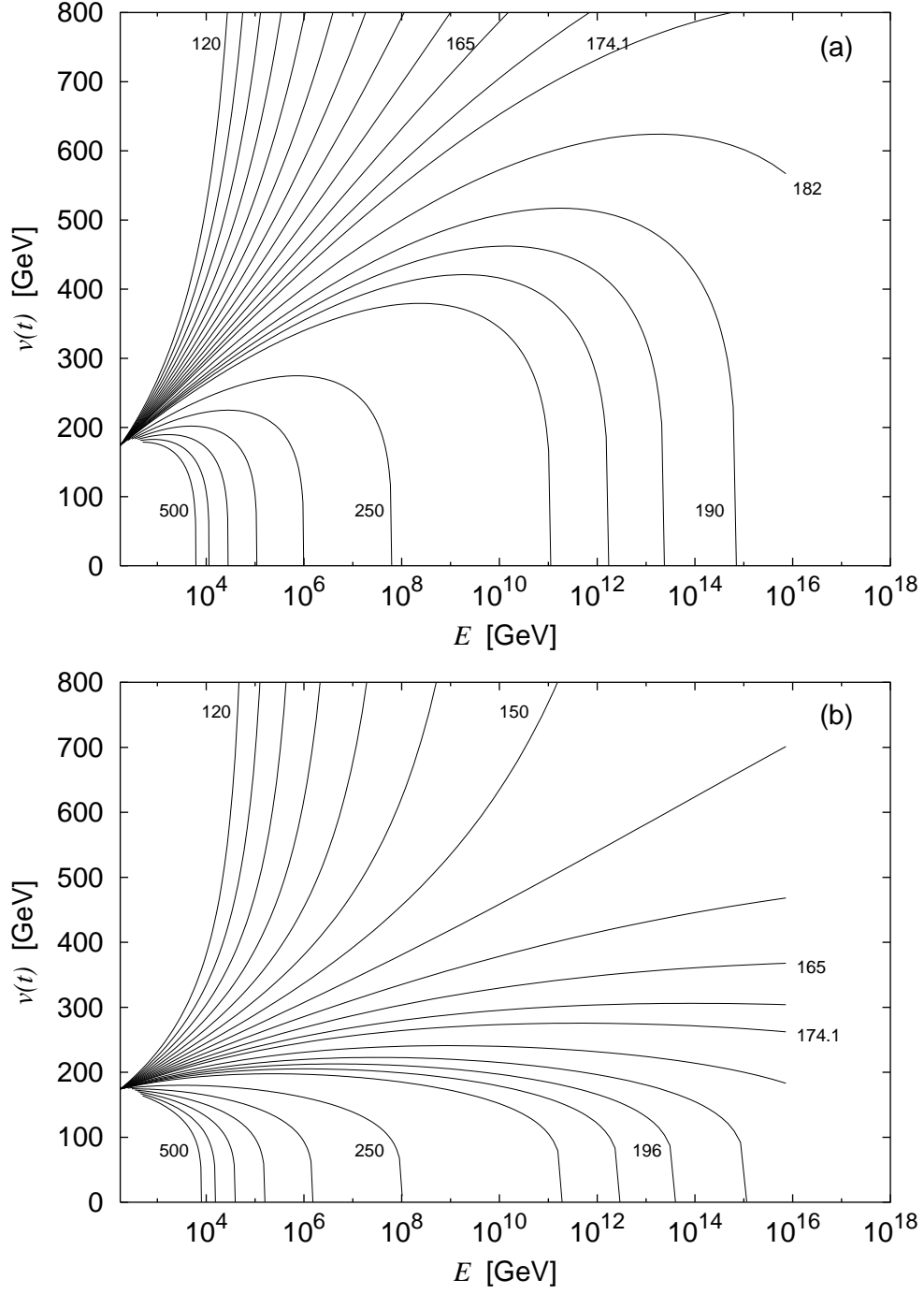


FIG. 6: Running of the vacuum expectation value of the Higgs field $v(t)$. Figure (a) corresponds to the case of the effective potential and figure (b) to the tree-level relation for the Higgs mass. The function $v(t)$ depends on the initial Higgs mass. The numbers at the ends of some plots correspond to the initial Higgs mass in GeV. In case (a) for the Higgs masses $m_H < 174$ GeV the $v(t)$ has a singularity at the UV cut off energy. For the Higgs masses $m_H > 190$ GeV the function $v(t)$ for large values of t is decreasing and at the UV cut off it vanishes. In case (b) for the Higgs masses $m_H < 164$ GeV the $v(t)$ has a singularity at the UV cut off energy. For the Higgs masses $m_H > 178$ GeV the function $v(t)$ for large values of t is decreasing and at the UV cut off it vanishes. The behavior of $v(t)$ in both cases is significantly different.

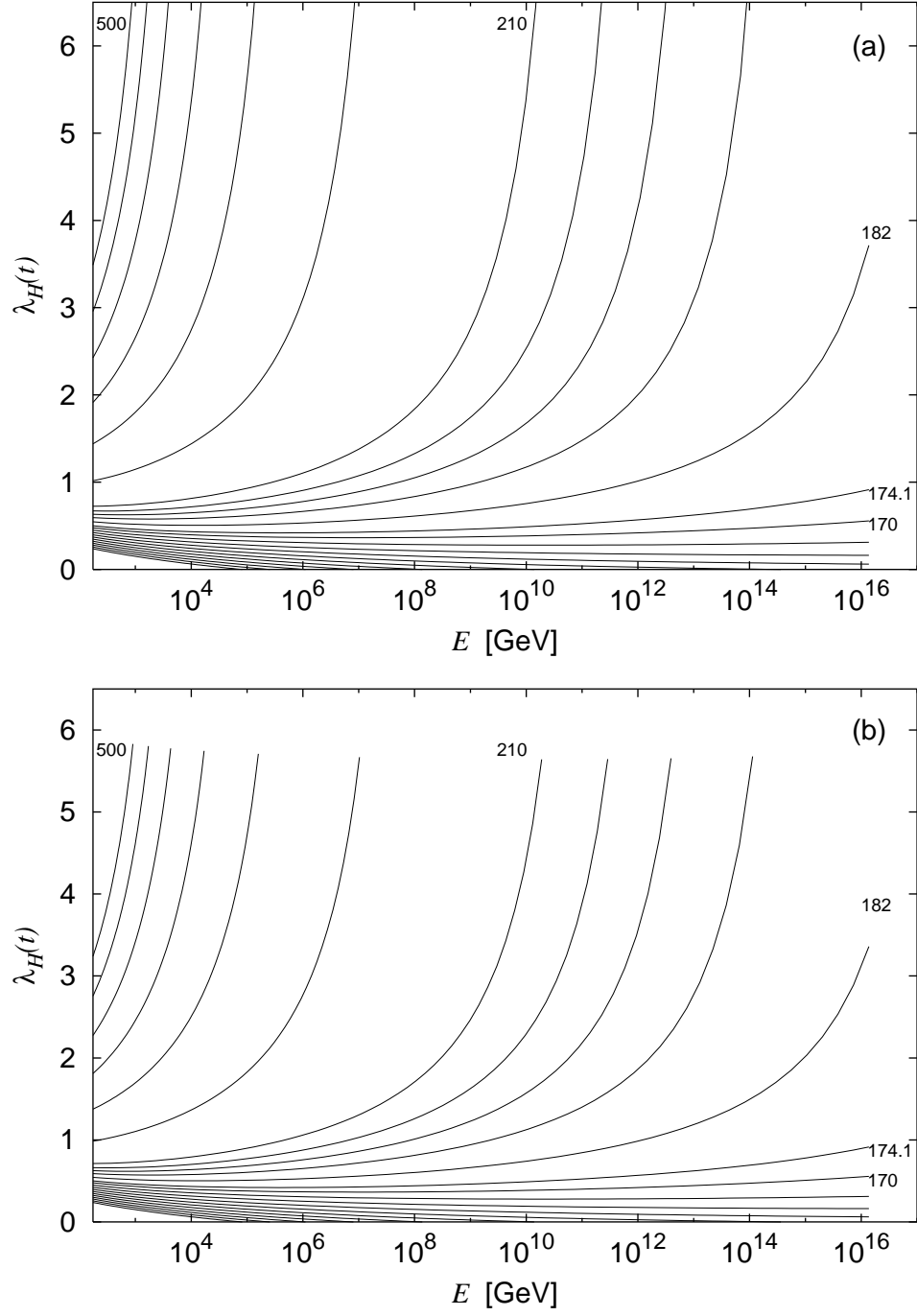


FIG. 7: Running of $\lambda_H(t)$. Figure (a) corresponds to the case of the effective potential and figure (b) to the tree-level relation for the Higgs mass. The function $\lambda_H(t)$ depends on the initial Higgs mass. The numbers at the ends of some plots correspond to the initial Higgs mass in GeV. One can see that for low values of $\lambda_H(0)$ which correspond to the low Higgs masses the function $\lambda(t)$ has a zero and for higher values of $\lambda_H(0)$ corresponding to larger Higgs masses there appears a Landau pole. Both cases (a) and (b) give very similar predictions.